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# Special Function Solutions of a Class of Certain Non-autonomous Nonlinear Ordinary Differential Equations

$K.M.Tamizhmani^{*}$, $K.Krishnakumar^{*}$, $A.Annamalai^{\$}$

**Abstract—** We apply a direct linearization technique to a class of certain nonlinear non-autonomous second-order ordinary differential equations with cubic nonlinearities and obtained special function solutions of them.

**Index Terms**— Nonlinear differential equations, Riccati equation and Special functions

## 1 Introduction

Finding solutions of nonlinear differential equations in terms of elementary functions or special functions has been an interesting area of research since 1860 [1], [2]. Very often, we come to a situation wherein such exact solvability is not possible. Though, an extensive works being carried out by many authors by employing various techniques including group of symmetries to find solutions of ordinary differential equations (ODE's), still a general method is yet to emerge. Sophus Lie [3] made an attempt to unify the then existing methods of finding solutions through symmetry approach. Indeed, he had proposed a necessary and sufficient condition for a given second-order nonlinear ordinary differential equations which can be transformed to a linear ordinary differential equations through symmetries. For the last many years Lie's approach has been used and interesting linearizable equations were obtained. In recent years, many authors investigated the integrability/linearizability of many nonlinear second-order ordinary differential equations through Prelle-Singer method [6], [7], [8], [9], [10], [11], Jacobi last multiplier method [12], [17], [18], symmetry approach [3] etc.. In this paper, we investigate the linearization of the nonlinear ODE's through Riccati equation. In our analysis, we never use either first integral or symmetries for linearization. It is well-known that Riccati equation

$$x'(t) = a(t)x(t)^2 + b(t)x(t) + c(t), \quad (1)$$

can be linearized through the Cole-Hopf transformation

$$x(t) = -\frac{y'(t)}{a(t)y(t)}, \quad (2)$$

to a linear second-order ODE of the form

$$y''(t) + p(t)y'(t) + q(t)y = 0, \quad (3)$$

where $p(t) = -b(t) + \frac{a'(t)}{a(t)}$, $q(t) = a(t)c(t)$.

Note that in the above process, the solution of the Riccati is expressed through the solution of the linear equation. In this paper, we propose to find a special class of solutions for many second-order nonlinear ordinary differential equations through Riccati equation.

## 2 Method

Consider the problem of finding special solutions of the second-order nonlinear ordinary differential equation of the form

* Department of Mathematics, Pondicherry University, Puducherry-605 014,
kmtmani54@gmail.com
krishapril09@gmail.com
$ Department of Mathematics, Government Arts College, Thiruvannamalai-606 603.

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$$x''(t) = f(x'(t), x(t), t), \quad (4)$$

where prime denotes the differentiation of $x$ with respect to '$t$'. Recently, in a series of papers [6], [7] the authors successfully found first integrals and obtained explicit solutions for a class of autonomous nonlinear ODE's. In [15], [16], Riccati equation was used to find special function solutions of Painlevé equations for specific parametric restrictions, a non-autonomous nonlinear ODE's. So, the natural question arises whether one could construct special function solutions of non-autonomous form equations considered in [6], [7]

In this paper, we give affirmative answer to this question by proposing a direct linearization procedure. Indeed, we find new linearizable time dependent systems. The solutions of them are expressed in terms of Bessel or elementary functions. Our strategy is the following. We assume that $x(t)$ is a solution for both (5) and Riccati equation (1) simultaneously

$$x'(t) = a(t)x(t)^2 + b(t)x(t) + c(t),$$

where $a(t)$, $b(t)$ and $c(t)$ are unknown functions to be determined. On substituting (1) into (4) one obtains an over-determined system of equations for $a(t)$, $b(t)$ and $c(t)$. On solving them consistently, one can easily find the values of the functions $a(t)$, $b(t)$ and $c(t)$, together with the parametric restrictions of (4), if there are any. It is evident from the numerous examples presented below that our approach is often simple, direct and can be readily applicable to a wide variety of problems, including higher-order nonlinear ODE's.

Our aim is to employ the above mentioned method to find explicit solutions for certain class of equations recently studied in Refs. [6], [7] and [10]. Our investigation reveals a new class of non-autonomous form of the equations studied in [6], [7] and furthermore, we obtain these equations admit new solutions.

## 3 Examples

We consider the following integrable equations considered by Chandrasekar et al. [6], [7] which are cubic in $x(t)$:

$$x''(t) + (k_1x(t) + k_2)x'(t) + k_3x(t)^3 + k_4x(t)^2 + \lambda\, x(t) = 0, \quad (5)$$

$$x''(t) + (k_1x(t) + k_2)x'(t) + \frac{k_1{}^2}{9}x(t)^3 + \frac{k_1k_2}{3}x(t)^2 + \lambda\, x(t) = 0, \quad (6)$$

$$x''(t) + (k_1x(t) + k_2)x'(t) + \frac{(r-1)k_1{}^2}{2\,r^2}x(t)^3 + \frac{k_1k_2}{3}x(t)^2 + \lambda\, x(t) = 0, \quad (7)$$

$$x''(t) + k_1x(t)x'(t) + k_3x(t)^3 + \lambda\, x(t) = 0, \quad (8)$$

$$x''(t) + (k_1x(t) + k_2)x'(t) + k_3x(t)^3 + \frac{k_1k_2}{3}x(t)^2 + \lambda\, x(t) = 0, \quad (9)$$

$$x''(t) + k_2x'(t) + k_4x(t)^2 + \lambda\, x(t) = 0, \quad (10)$$

$$x''(t) + (k_1x(t) + k_2)x'(t) + \frac{k_1k_2}{3}x(t)^3 + \lambda\, x(t) = 0, \quad (11)$$

where $k_1, k_2, k_3, k_4$ *and* $\lambda$ are arbitrary constants.

## 4 Examples

### Example 1

Now, we generalize the above systems by considering $\lambda$ as a function of 't'. Thus, we have

$$x''(t) + (k_1x(t) + k_2)x'(t) + k_3x(t)^3 + k_4x(t)^2 + \lambda(t)\, x(t) = 0, \quad (12)$$

We apply the linearization procedure outlined above for (12). Differentiate (1) and equate with (12), we get an over-determined system of equations for the unknowns. In this case, we have the following conditions:

$$k_3 + k_1a(t) + 2a(t)^2 = 0\,, \quad (13)$$

$$k_4 + k_2a(t) + k_1b(t) + 3a(t)b(t) + a'(t) = 0, \quad (14)$$

$$c(t) + b(t)a(t) + c'(t) = 0\,, \quad (15)$$

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$$\lambda(t) + b(t) + b(t)^2 + k_1 c(t) + 2a(t)c(t) + b'(t) = 0 \,. \qquad (16)$$

Solving (13), we get

$$a(t) = \frac{1}{4}\left(-k_1 \pm \sqrt{k_1^2 - 8k_3}\right) = \chi \text{ (say)}. \quad (17)$$

Substitute (17) into (14) then we find

$$b(t) = \frac{-k_2\chi - k_4}{3\chi + k_1} = \omega \text{ (say)}. \qquad (18)$$

Using (18) in (15) we arrive at

$$c(t) = c_1 e^{-2\mu t}, \qquad (19)$$

where $2\mu = \omega + k_2$ and $c_1$ is a constant of integration. Again, we use the values of $a(t)$, $b(t)$ and $c(t)$ in (16) and find

$$\lambda(t) = -\omega^2 - \omega\ k_2 - c_1(2\chi + k_1)e^{-2\mu t}. \quad (20)$$

Therefore, we finally arrive at the following Riccati equation:

$$x'(t) = \chi\, x(t)^2 + \omega x(t) + c_1 e^{-2\mu t}. \qquad (21)$$

Now, (21) can be linearized through Cole-Hopf type of transformation

$$x(t) = -\frac{y'(t)}{\chi\, y(t)}. \qquad (22)$$

Substituting (22) into (21) one finds, a second-order linear ordinary differential equation of the form

$$y''(t) - \omega\, y'(t) + c_1\, \chi e^{-2\mu t} y(t) = 0. \, (23)$$

Introducing the change of variable $z = \pm\sqrt{c_1\, \chi}\ e^{-\mu t}$ in (23), we arrive at the following Bessel type of equation

$$y''(z) + \left(1 + \frac{\omega}{\mu}\right) z y'(z) + \frac{z^2}{\mu^2} y(z) = 0. \quad (24)$$

(24) is a special case of a Bessel's equation given in [4] (Eq.(6.88) pp.280). Thus, (12) admits Bessel's function solution. Rational solutions follows from (22). Now, we specialize different cases according to different choices of the parameters.

## Example 2

If $k_3 = \frac{k_1^2}{9}$ and $k_4 = \frac{k_1 k_2}{3}$ then the non-autonomous form of (6) is given by

$$x''(t) + (k_1 x(t) + k_2)x'(t) + \frac{k_1^2}{9} x(t)^3 + \frac{k_1 k_2}{3} x(t)^2 + \lambda(t)x(t) = 0. \qquad (25)$$

On using the above parametric restrictions in (13), (14), (15) and (16) one arrive at two set of values for $a(t)$, $b(t)$, $c(t)$ and $\lambda(t)$.

Case (i):

Consider $a(t) = -\frac{k_1}{3} = \chi$ (say), $b(t) = \omega$ (arbitrary), $c(t) = c_1 e^{-2\mu t}$, $\lambda(t) = -\omega^2 - \omega\ k_2 - c_1 k_1 e^{2\mu t}$ , where $2\mu = k_2 + \omega$. Now the Riccati equation assumes the form

$$x'(t) = -\frac{k_1}{3}\, x(t)^2 + \omega x(t) + c_1 e^{-2\mu t}. \qquad (26)$$

Though (26) is same form as (21) but with the values of $\chi$ and $\omega$ as given above. Hence, it is obvious again to conclude that this Riccati also can be identified with Bessel's equation. Thus, (25) admits Bessel's function solution as well.

Case (ii):

Now, consider the case $a(t) = -\frac{k_1}{6} = \chi$ (say), $b(t) = -\frac{k_2}{3} = \omega$ (say), $c(t) = c_1 e^{2\omega t}$ and $\lambda(t) = 2\omega^2 + 4c_1\ \chi\ e^{2\omega t}$ . The corresponding Riccati equation now becomes

$$x'(t) = \chi\, x(t)^2 + \omega x(t) + c_1 e^{2\omega t}. \qquad (27)$$

Using the Cole-Hopf type of transformation (2)

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in the Riccati (27), we obtain a second-order linear ordinary differential equation of the form

$$y''(t) - \omega\, y'(t) + c_1\ \chi e^{2\omega t}\ \ y(t) = 0. \quad (28)$$

Introducing the change of variable $z = \pm\sqrt{c_1\chi}\ e^{\omega t}$ in (28), we arrive at

$$y''(z) + \frac{1}{\omega^2} y(z) = 0\,. \quad (29)$$

Solving (29), we get

$$y(z) = c_2 \cos\beta z + c_3\ \sin\beta z, \quad (30)$$

where $\beta = \frac{1}{\omega}$. Substituting (32) in (2), then we get the solution of (25) as

$$x(t) = \pm\sqrt{\frac{c_1}{\chi}}\, e^{\omega t}\left(\frac{c_3\cos(\beta\sqrt{c_1\chi}\, e^{\omega t}) \mp c_2 \sin(\beta\sqrt{c_1\chi}\, e^{\omega t})}{c_2\cos(\beta\sqrt{c_1\chi}\, e^{\omega t}) \pm c_3 \sin(\beta\sqrt{c_1\chi}\, e^{\omega t})}\right). \quad (31)$$

Equation (31) can be rewritten as

$$x(t) = \pm\sqrt{\frac{c_1}{\chi}}\, e^{\omega t}\left(\frac{1 \mp \delta\, tan\,(\beta\sqrt{c_1\chi}\, e^{\omega t})}{\delta \pm tan\,(\beta\sqrt{c_1\chi}\, e^{\omega t})}\right), \quad (32)$$

where $\beta = \frac{1}{\omega}$ and $\delta = \frac{c_2}{c_3}$.

Example 3

Now we consider non-autonomous form of (7)

$$x''(t) + (k_1 x(t) + k_2) x'(t) + \frac{k_1 k_2}{3} x(t)^2 + \frac{(r-1)k_1^2}{2r^2} x(t)^3 + \lambda(t) x(t) = 0\,. \quad (33)$$

Case (i):

Consider $a(t) = -\frac{k_1}{2r} = \chi$ (say), $b(t) = -\frac{k_2}{3} = \omega$ (say), $c(t) = c_1 e^{2\omega t}$ and $\lambda(t) = \frac{2k_2^2}{9} + \left(r + \frac{1}{r}\right) k_1 c_1 e^{2\omega t}$, where $r \neq \frac{3}{2}$. In this case, the Riccati equation becomes

$$x'(t) = \chi\, x(t)^2 + \omega x(t) + c_1 e^{2\omega t}\,, \quad (34)$$

whose solution is

$$x(t) = \pm\sqrt{\frac{c_1}{\chi}}\, e^{\omega t}\left(\frac{c_3\cos(\beta\sqrt{c_1\chi}\, e^{\omega t}) \mp c_2 \sin(\beta\sqrt{c_1\chi}\, e^{\omega t})}{c_2\cos(\beta\sqrt{c_1\chi}\, e^{\omega t}) \pm c_3 \sin(\beta\sqrt{c_1\chi}\, e^{\omega t})}\right). \quad (35)$$

In other words

$$x(t) = \pm\sqrt{\frac{c_1}{\chi}}\, e^{\omega t}\left(\frac{1 \mp \delta\, tan\,(\beta\sqrt{c_1\chi}\, e^{\omega t})}{\delta \pm tan\,(\beta\sqrt{c_1\chi}\, e^{\omega t})}\right), \quad (36)$$

where $\beta = \frac{3}{k_2}$ and $\delta = \frac{c_2}{c_3}$. If $r = \frac{3}{2}$ then (33) is same as (25).

Case (ii):

Now, we take $a(t) = -\frac{(1-r)k_1}{2r} = \chi$ (say). In this case, we find $b(t) = -\frac{k_2}{3} = \omega$ (say), $c(t) = c_1 e^{2\omega t}$ and $r \neq 3$. Here, we obtain the Riccati equation

$$x'(t) = \chi\, x(t)^2 + \omega x(t) + c_1 e^{2\omega t}, \quad (37)$$

whose solution is

$$x(t) = \pm\sqrt{\frac{c_1}{\chi}}\, e^{\omega t}\left(\frac{c_3\cos(\beta\sqrt{c_1\chi}\, e^{\omega t}) \mp c_2 \sin(\beta\sqrt{c_1\chi}\, e^{\omega t})}{c_2\cos(\beta\sqrt{c_1\chi}\, e^{\omega t}) \pm c_3 \sin(\beta\sqrt{c_1\chi}\, e^{\omega t})}\right). \quad (38)$$

Which is nothing but

$$x(t) = \pm\sqrt{\frac{c_1}{\chi}}\, e^{\omega t}\left(\frac{1 \mp \delta\, tan\,(\beta\sqrt{c_1\chi}\, e^{\omega t})}{\delta \pm tan\,(\beta\sqrt{c_1\chi}\, e^{\omega t})}\right), \quad (39)$$

where $\beta = \frac{1}{\omega}$ and $\delta = \frac{c_2}{c_3}$. If $r = 3$ then (33) is same as (25).

Example 4

Our next example is the non-autonomous form of (8):

$$x''(t) + k_1 x(t) x'(t) + k_3 x(t)^3 + \lambda(t) x(t) = 0, \quad (40)$$

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,

where $k_1$ and $k_3$ are non zero. From (13), (14), (15) and (16) we find

$$a(t) = \frac{1}{4}\left(-k_1 \pm \sqrt{k_1^2 - 8k_3}\right) = \chi \text{ (say)}, \quad (41)$$

$$b(t) = 0, \quad (42)$$

$$c(t) = c_1, \quad (43)$$

where $c_1$ is a constant of integration. Using (41), (42) and (43) in (16), we find that

$$\lambda = c_1(2\chi - k_1). \quad (44)$$

Thus, the Riccati equation associated with (1) is given in the form

$$x'(t) = \chi\, x(t)^2 + c_1. \quad (45)$$

Equation (45) can be linearizable through

$$x(t) = -\frac{y'(t)}{\chi\, y(t)} \quad (46)$$

and obtain

$$y''(t) + c_1\chi\, y(t) = 0. \quad (47)$$

The solution of (47) is

$$y(t) = c_2 \cos(\sqrt{c_1\chi}\, t) + c_3 \sin(\sqrt{c_1\chi}\, t), \quad (48)$$

where $\chi = \frac{1}{4}\left(-k_1 \pm \sqrt{k_1^2 - 8k_3}\right)$. Substituting (48) in (46), we obtain the solution

$$x(t) = \sqrt{\frac{c_1}{\chi}}\left(\frac{c_2 \sin(\beta\sqrt{c_1\chi}\, t) - c_3 \cos(\beta\sqrt{c_1\chi}\, t)}{c_2 \cos(\beta\sqrt{c_1\chi}\, t) + c_3 \sin(\beta\sqrt{c_1\chi}\, t)}\right). \quad (49)$$

Now, (49) can be rewritten as

$$x(t) = \sqrt{\frac{c_1}{\chi}}\left(\frac{-1 + \delta \tan(\beta\sqrt{c_1\chi}\, t)}{\delta + \tan(\beta\sqrt{c_1\chi}\, t)}\right), \quad (50)$$

where $\delta = \frac{c_2}{c_3}$, which is a solution of (40).

Example 5

In [7], now we consider the non-autonomous equation of equation given in [7]

$$x''(t) + (k_1 x(t) + k_2)x'(t) + k_3 x(t)^3 + \frac{k_1 k_2}{3} x(t)^2 + \lambda(t)x(t) = 0. \quad (51)$$

By repeating the same steps carried out in the preceding examples, we arrive at $a(t) = \frac{1}{4}\left(-k_1 \pm \sqrt{k_1^2 - 8k_3}\right) = \chi$ (say), $b(t) = -\frac{k_2}{3} = \omega$ (say), $c(t) = c_1 e^{2\omega t}$ and $\lambda(t) = \frac{2k_2^2}{9} - (2\chi + k_1)c_1 e^{2\omega t}$. Note that when $k_1 = -2\chi$, $\lambda(t)$ reduced to the case given in [4]. Therefore, our choice of $\lambda(t)$ gives a more general form of linear equation. For this new linearizable equation we present explicit solution. For these values, we get Riccati equation of the form

$$x'(t) = \chi\, x(t)^2 + \omega x(t) + c_1 e^{2\omega t}, \quad (52)$$

which can be linearized through Cole-Hopf transformation (2). Thus, we get a second-order linear ordinary differential equation

$$y''(z) + \frac{1}{\omega^2} y(z) = 0. \quad (53)$$

Equation (53) same form as (29) but with different values of $\chi$ and $\omega$. The solution of (52) can be expressed as

$$x(t) = \sqrt{\frac{c_1}{\chi}}\, e^{\omega t}\left(\frac{c_3 \cos(\beta\sqrt{c_1\chi}\, e^{\omega t}) \mp c_2 \sin(\beta\sqrt{c_1\chi}\, e^{\omega t})}{c_2 \cos(\beta\sqrt{c_1\chi}\, e^{\omega t}) \pm c_3 \sin(\beta\sqrt{c_1\chi}\, e^{\omega t})}\right). \quad (54)$$

This implies

$$x(t) = \pm\sqrt{\frac{c_1}{\chi}}\, e^{\omega t}\left(\frac{1 \mp \delta \tan(\beta\sqrt{c_1\chi}\, e^{\omega t})}{\delta \pm \tan(\beta\sqrt{c_1\chi}\, e^{\omega t})}\right), \quad (55)$$

where $\beta = \frac{1}{\omega}$ and $\delta = \frac{c_2}{c_3}$.

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Example 6

Now, we consider the non-autonomous form of (10)

$$x'' + k_2\, x' + k_4 x(t)^2 + \lambda(t)x(t) = 0. \quad (56)$$

We get the following Riccati equation

$$x'(t) = \chi\, x(t)^2 + \omega x(t) + c_1 e^{\omega t}\,, \quad (57)$$

and the second-order linear differential equation is given that

$$y''(t) - \omega\, y'(t) + c_1\, \chi e^{\omega t}\, y(t) = 0, \quad (58)$$

where $\chi = \pm\sqrt{\frac{-k_3}{2}}$, $\omega = \mp\frac{k_4}{3\omega}$, $c(t) = c_1 e^{-\omega t}$ and $\lambda = \frac{2k_4^2}{9k_3} - 2c_1\omega e^{\omega t}$. (58) can be rewritten as

$$z^2\frac{d^2y}{dz^2} + 3z\frac{dy}{dz} + \frac{4z^2}{\omega^2}y(z) = 0, \quad (59)$$

where $z = \pm\sqrt{c_1\chi}\, e^{-\frac{\omega t}{2}}$. It is again the Bessel's type of equation. Thus, (56) admits Bessel function solution.

Example 7

Finally, we consider the non-autonomous form of (11)

$$x''(t) + (k_1 x(t) + k_2)x'(t) + \frac{k_1 k_2}{3}x(t)^3 + \lambda(t)x(t) = 0. \quad (60)$$

For this equation, we can get the Riccati equation and the corresponding second-order linear differential equation as follows:

$$x'(t) = \chi\, x(t)^2 + \omega x(t) + c_1 e^{-(\omega+k_2)t} \quad (61)$$

and

$$y'' - \omega\, y' + c_1\chi e^{-(\omega+k_2)t}y(t) = 0, \quad (62)$$

where $\chi = \frac{1}{4\sqrt{3}}\left(-\sqrt{3}k_1 \pm \sqrt{3k_1^2 - 8k_1k_2}\right)$, $\omega = -\frac{k_2\chi}{3\chi+k_1}$ and $\lambda = -\omega^2 - k_2\omega - (2\chi + k_1)c_1 e^{-(\omega+k_2)t}$. Through a change of variable $z = \sqrt{c_1\chi}\, e^{-(\frac{\omega+k_2}{2})t}$. (62) becomes

$$z^2\frac{d^2y}{dz^2} + \left(1 + \frac{2\omega}{\omega+k_2}\right)z\frac{dy}{dz} + \frac{4z^2}{(\omega+k_2)^2}y(z) = 0, \quad (63)$$

which is again a Bessel's type of equation. Therefore, the solution can be expressed in terms of Bessel function.

## 5 Conclusions

In this paper, we have presented particular solutions for certain class of nonlinear non-autonomous second-order ordinary differential equations with cubic nonlinearity through direct linearization procedure. We have obtained a large class of new solutions for these equations in terms of Bessel and elimentary functions. Our future plan is to extend this analysis to second-order nonlinear differential equations with high-er nonlinearity, coupled systems and higher-order equations.

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## References

[1] A.L. Cauchy, "*Mémoire sur l'intégration des équations différentielles*" 1835.
[2] A.L. Cauchy, "Mémoire sur l'intégration des équations différentielles des mouvements planétaires." pp.184-190, *CRC* 9 (1839).
[3] S. Lie, "On Integration of a Class of Linear Partial Differential Equations by Means of Definite Integrals." *Arch. Math VI (3)* 1.88 (1881).
[4] L.C. Andrews, "Special Functions of Mathematics for Engineers', Oxford, USA (1998).
[5] A.M. Grundland and D. Levi, "On Higher-order Riccati Equations as Backland Transformations', J. Phys. A: Math. Gen. 32, pp.3931-3937 (1999).
[6] V.K. Chandrasekar, S.N. Pandey, M. Senthilvelan, and M. Lakshmanan, "A Simple and Unified Approach to Identify Integrable Nonlinear Oscillators and Systems", J. Math. Phys. 47, 023508, (2006).

International Journal of Scientific & Engineering Research Volume 6, Issue 3, March-2015
ISSN 2229-5518

[7] V.K. Chandrasekar, S.N. Pandey, M. Senthilvelan, and M. Lakshmanan, "Extended Prelle-Singer Method and Integrability/Solvability of a Class of Nonlinear n-th Order Ordinary Differential Equations", J. Non- linear Math. Phys. 12, pp.184-201 (2005).

[8] V.K. Chandrasekar, M. Senthilvelan, Anjan Kundu and Lakshmanan M., "A Nonlocal Connection Between Certain Linear and Nonlinear Ordinary Differential Equations/Oscillators", J. Phys. A: Math. Gen. 39, pp.9743-9754 (2006).

[9] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, "On the Complete Integrability and Linearization of Nonlinear Ordinary Differential Equations. II. Third-Order Equations", Proc. R. Soc. London A, 732, pp.1831-1852(2006).

[10]Chandrasekar V.K., M. Senthilvelan and M. Lakshmanan, "On the Complete Integrability and Linearization of Certain Second-Order Nonlinear Ordinary Differential Equations", Proc. R. Soc. London A, 461, pp.2451-2477(2005).

[11] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, "Unusual Liénard-Type Nonlinear Oscillator", Phys. Rev. E 72, 066203(2005).

[12]M.C. Nucci and K.M. Tamizhmani, "Lagrangians for Dissipative Nonlinear Oscillators: the Method of Jacobi Last Multiplier", J. Nonlinear Math. Phys. 17, pp.167-178(2009).

[13] T. Tamizhmani, B. Grammaticos, A. Ramani and K.M. Tamizhmani, "On a Class of Special Solutions of the Painlevé Equations", Physica A 295, pp.359-370 (2001).

[14] Tamizhmani T., B. Grammaticos, K.M. Tamizhmani and A. Ramani, "Special Solutions of Discrete Painlevé Equations Through Direct Linearization", Physica A 315, pp.569-582 (2002).

[15] A. Ramani, B. Grammaticos, T. Tamizhmani and K.M. Tamizhmani, "Special Function Solutions of the Discrete Painlevé Equations", Computers and Mathematics with Applications 42, pp.603-614 (2001).

[16] V.A. Gromak and N.A. Lukashevich., "The Analytical Solutions of the Painlevé Equations, Minsk (1990, in Russian).

[17] M.C. Nucci and P.G.L. Leach, "The Jacobi Last Multiplier and Its Application in Mechanics", Phys. Scr. 78, pp.1-6(2008).

[18] M.C. Nucci and P.G.L. Leach, "Jacobi's Last Multiplier and Lagrangians for Multidimensional Systems", J. Math. Phys. 49, pp.1-8 (2008).